Relation between size and shape of isotropic and directed percolation clusters and lattice animals

LETTER TO THE EDITOR

Relation between size and shape of isotropic and directed percolation clusters and lattice animals

Fereydoon Family

Physics Department, Emory University, Atlanta, GA 30322, USA

Abstract. We show that the generalised lattice animal model of Family and Coniglio naturally leads to a unified scaling picture for percolation and lattice animals in which the fugacity for occupied elements plays the dual role of a temperature-like and a field-like variable. Within this single-scaling-field description of percolation, there is only one independent exponent from which all others can be obtained. We define a new set of exponents α, β and γ for percolation and find that they are all related to the cluster number exponent θ through the relation $\alpha = \gamma = 1 - \beta = 3 - \theta$, in analogy with lattice animals. To relate the cluster radius exponent ν to the other exponents we use the generalised Ginzburg criteria to obtain a modified hyperscaling relation for isotropic and directed, percolation and lattice animals. Using this relation we find that $\theta - 1 = \nu_{\parallel} + \nu_{\perp}(d-1)$ for directed percolation and $\theta = \nu_{\perp}(d-1)$ for directed lattice animals, where $\nu_{\parallel}$ and $\nu_{\perp}$ are exponents characterising the parallel and perpendicular cluster radii respectively. Using the same approach we obtain the Stauffer relation $\theta - 1 = d\nu$ and the Parisi–Sourlas relation $\theta - 1 = (d-2)\nu$ for isotropic percolation and lattice animals respectively. The above relations give the following expressions for θ within the Flory theory: $\theta(\text{percolation}) = (3d+2)/(d+2)$, $\theta(\text{directed percolation}) = (6d+5)/(2d+4)$, $\theta(\text{animals}) = (7d-6)/(2d+4)$ and $\theta(\text{directed animals}) = 9(d-1)/(4d+8)$.

1. Introduction

Percolation and lattice animals have been subjects of considerable recent interest because of their applications to many fields of physics (see e.g. Stauffer (1979) and Essam (1980) for recent reviews), and in particular to polymers (see e.g. Stauffer *et al* (1982) and Stanley *et al* (1982) for recent reviews of applications to polymers). More recently a special variation of these models—called directed percolation and directed animals—in which bonds are oriented with respect to only certain preferred directions has also received considerable attention (Blease 1977 a, b, c, Kertész and Vicsek 1980, Reynolds 1980, Cardy and Sugar 1980, Obukhov 1980, Dhar and Barma 1981, Essam and De'Bell 1981, Kinzel and Yeomans 1981, Klein and Kinzel 1981, Redner and Reynolds 1981, Wu and Stanley 1982, Redner and Yang 1982, Dhar *et al* 1982, Day and Lubensky 1982, Lubensky and Vannimenus 1982.) The main reasons for the interest in the directed models is that they belong to a different universality class (Reynolds 1980) from isotropic models and they have application to a wide variety of problems, including Reggeon field theory (Cardy and Sugar 1980), Markov processes involving branching (Schlögl 1972), and dilute branched polymers in flowing solvents.

In the usual approach to percolation most of the critical exponents are defined in the limit where the probability p approaches its critical value p_c, i.e. an exponent ω for a quantity X is defined by $X \sim |p_c - p|^\omega$. However, in cluster models one can alternatively follow the convention used in the scaling theory of polymers (see de Gennes (1979) and references therein) of defining sets of exponents in terms of the cluster size N directly. The two most commonly studied cluster properties are the cluster radius and the cluster number C_N which, as $N \to \infty$, have the scaling forms

$$\xi \sim N^\nu \tag{1}$$

and

$$C_N \sim N^{-\theta} \lambda^N \tag{2}$$

where ν and θ are the cluster radius and cluster number exponents† respectively. The quantity λ is a non-universal lattice-dependent parameter. In directed percolation and directed lattice animals the structure of large clusters is highly anisotropic and must be characterised by two length scales $\xi_\parallel$ and $\xi_\perp$, one parallel and one perpendicular to the directed axis (Dhar and Barma 1981, Kinzel and Yeomans 1981). Divergence of these length scales is characterised by two exponents $\nu_\parallel$ and $\nu_\perp$ respectively, which are defined by

$$\xi_\parallel \sim N^{\nu_\parallel} \tag{3}$$

and

$$\xi_\perp \sim N^{\nu_\perp}. \tag{4}$$

Although considerable effort has recently been spent in calculations of the above exponents, relatively little is known in the way of a relation between the exponents ν and θ for each model, particularly for directed percolation and directed animals. In this Letter we develop a single-scaling-field approach to percolation to obtain relations between the critical exponents of directed percolation and directed lattice animals, using scaling arguments and the Ginzburg criteria, and show that the relations obtained earlier by Stauffer (see e.g. Stauffer (1979) and references therein) and by Parisi and Sourlas (1980) for isotropic percolation and isotropic lattice animals respectively, follow directly from these arguments. In particular, the Parisi–Sourlas relation is obtained without reference to the Lee–Yang edge singularity.

2. Single-scaling-field approach for percolation: generalised lattice animals

Recently, Family and Coniglio (1980), and Family and Reynolds (1981) have introduced a generalisation of percolation and lattice animals by characterising an arbitrary cluster by two independent parameters: a fugacity K for each occupied element in the cluster and an independent fugacity q for each perimeter site or bond. The generating function for generalised lattice animals is written as (Family and Coniglio 1980)

$$Z(K, q) = \sum_N D_N(q) K^N \tag{5}$$

---

† In the percolation model (Stauffer 1979, Essam 1980) ν and θ are called ρ and τ respectively.

where the perimeter polynomials are defined by

$$D_N(q) = \sum_{N_p} C(N, N_p) q^{N_p}. \tag{6}$$

Here $C(N, N_p)$ are the number of distinct clusters of size N and perimeter size N_p. In analogy with the scaling relation (2) we assume that at fixed q in the $N \to \infty$ limit, $D_N(q)$ varies as

$$D_N(q) \sim N^{-\theta(q)} \lambda^N(q) \qquad (N \to \infty) \tag{7}$$

so that $Z(K, q)$ as a function of K has a singular behaviour of the form

$$Z(K, q) = |K_c(q) - K|^{\theta(q)-1} \tag{8}$$

where $K_c(q) = \lambda(q)^{-1}$ is the critical point. Note that $\lambda(1)^{-1} = K_c$ is the critical fugacity for random animals and $\lambda(q_c)^{-1} = p_c$ is the percolation threshold. In the crossover region (finite, small N) an extra scaling function must be introduced in (7) (see e.g. Stauffer 1979).

It is not difficult to show (Family and Reynolds 1981) that $N \sim |K_c(q) - K|^{-1}$, so that K and N are conjugate variables, and every exponent defined through relations such as (1)–(4) along the path $N \to \infty$ is uniquely related to an exponent along the path $K \to K_c$.

Although K plays the role of a temperature-like variable, here, it may also be regarded as a field variable (for an application of this concept to the self-avoiding-walk problem see Redner and Reynolds (1981) and Havlin and Ben–Avraham (1982)) because it couples to every element in a cluster and plays the same thermodynamic role as an external ghost field for the system. This dual role played by K is an important property of most random cluster models in which the scaling properties depend only on N. Thus, in contrast to previous approaches to percolation which define a 'temperature' scaling power y_p and a 'magnetic' scaling power y_h, in the present approach there exists only a single scaling power in percolation and all the critical exponents are related.

In order to define the new critical exponents we assume that, in analogy with percolation, $Z(K, q)$ plays the role of a Gibbs potential. Thus the 'order parameter' M, which is obtained by differentiating the Gibbs potential with respect to the field-like variable, has the asymptotic form

$$M(K, q) \sim |K_c(q) - K|^{\theta(q)-2}. \tag{9}$$

The 'susceptibility' χ is in turn determined by differentiating the order parameter with respect to the field:

$$\chi(K, q) \sim |K_c(q) - K|^{\theta(q)-3}. \tag{10}$$

Recall the usual definitions of the critical exponents α, β and γ:

$$Z \sim |K_c - K|^{2-\alpha} \qquad M \sim |K_c - K|^{\beta} \qquad \chi \sim |K_c - K|^{-\gamma}. \tag{11}$$

In comparing (8), (9) and (10) with (11), we find

$$\alpha(q) = \gamma(q) = 1 - \beta(q) = 3 - \theta(q) \tag{12}$$

which coincides with the definition and the relation between α, β, γ and θ for lattice animals (Lubensky and Isaacson 1979) if $q = 1$. At $q_c = 1 - p_c$, (12) defines a *new* set of exponents for percolation clusters as a function of cluster size N, and these exponents

are not the same as the usual percolation exponents α_p, β_p and γ_p (see e.g. Stauffer 1979 and Essam 1980), except $\theta(q_c) = \tau_p$. Although α, β, γ and θ are functions of q, according to universality, they cannot depend continuously on q. In fact, renormalisation group results (Family and Coniglio 1980, Harris and Lubensky 1981) imply that θ is a discontinuous function of the form

$$\theta(q) = \begin{cases} \theta(1) = \theta_{\text{animals}} & (1 \geqslant q > q_c) \\ \theta(q_c) = \theta_{\text{percolation}} & (q_c = 1 - p_c = 1 - K_c) \end{cases} \tag{13}$$

so that $\alpha(q)$, $\beta(q)$ and $\gamma(q)$ are also discontinuous functions of q.

3. Ginzburg criteria and modified hyperscaling relation

Since α, β and γ are all related to θ, if the hyperscaling relation $d\nu = 2 - \alpha$ could be assumed to hold then all exponents would be related. However, it is not *a priori* obvious that hyperscaling should be valid here. Therefore, we use a generalised Ginzburg-type argument (see e.g. Als-Nielsen and Birgeneau 1977) to arrive at an extended hyperscaling relation for both isotropic and directed, percolation and lattice animals.

According to the Ginzburg criteria the fluctuations of the order parameter M averaged over the critical volume Ω must be small compared with M. Thus, the general form of the Ginzburg criteria may be expressed as

$$\chi < \Omega M^2 \tag{14}$$

where χ is the susceptibility. As pointed out above, Ω is the volume over which critical fluctuations are important. Since ξ is the only relevant length in the problem, quite generally one can assume $\Omega \sim \xi^{d+m}$, where d is the dimensionality and m is a constant. Even though for many systems $m = 0$, there are a wide variety of important physical systems in which $m \neq 0$ (Als-Nielsen and Birgeneau 1977). Thus, using (11) in (14) and letting $\Omega \sim \xi^{d+m}$, the Ginzburg criteria may be written as

$$t^{-\gamma} < t^{-(d+m)\nu} t^{2\beta} \tag{15}$$

where $t = |K_c - K|/K_c$ in the present context. In order for (15) to be fulfilled, it is required that

$$\gamma - (d + m)\nu + 2\beta < 0. \tag{16}$$

The usual hyperscaling relation $d\nu = 2 - \alpha$ for isotropic thermal critical phenomena can be obtained from (16) by letting $m = 0$, using the scaling relation $2 - \alpha = \gamma + 2\beta$, and assuming that, since the inequality in (16) becomes an equality at the upper critical dimension d_c, where α, β, γ and ν take on their mean-field (MF) values, then the same relation holds for all $d \leqslant d_c$. Similarly, for isotropic systems, we define a modified hyperscaling relation

$$2 - \alpha = \nu(d + m) \tag{17}$$

by assuming an equality in (16) with an unknown parameter m to be determined self-consistently at d_c with the MF value of the exponents.

For directed percolation and directed animals (17) is further modified, because in these models there are two relevant length scales, $\xi_\parallel$ and $\xi_\perp$. This implies that the critical volume must generally be assumed to be of the form

$$\Omega \sim \xi_\parallel^{m+1}\xi_\perp^{d-1} \tag{18}$$

instead of $\Omega \sim \xi^{d+m}$. Using (18) in (14) we find that for directed percolation and animals

$$2 - \alpha = (m+1)\nu_\parallel + (d-1)\nu_\perp. \tag{19}$$

4. Isotropic percolation

A relation between θ and ν for isotropic percolation was obtained earlier by Stauffer (see e.g. § 4.2 in Stauffer (1979) and references therein) through a scaling assumption for the percolation cluster radius. Here we demonstrate our approach by obtaining this Stauffer relation.

The upper critical dimension d_c for percolation is 6 and in the MF approximation $\theta = \frac{5}{2}$ and $\nu = \frac{1}{4}$ (see e.g. Stauffer *et al* 1982). From (12) we find that in the MF approximation $\alpha = \beta = \gamma = \frac{1}{2}$. On substituting these results in (17) we find that $m = 0$ and the usual hyperscaling relation $d\nu = 2 - \alpha$ is valid for percolation clusters. Noting that $\alpha = 3 - \theta$ we arrive at the following exponent relation for isotropic percolation (see footnote below equation (5)):

$$\theta - 1 = d\nu. \tag{20}$$

We have used (20) and the known values of θ to determine ν in $d = 2$ to 6 and the results are given in table 1. Using the Flory approximation $\nu = 2/(d+2)$ (Isaacson and Lubensky 1980), for percolation we find $\theta = (3d+2)/(d+2)$.

Table 1. Isotropic percolation cluster radius exponent ν determined from known values of θ using (20). Other numerical estimates of ν are given in parentheses.

d	θ	$\nu\,(=(\theta-1)/d)$
2	$187/91 = 2.054\ldots$[a]	$48/91 = 0.527\ldots$
		$(0.53 \pm 0.01$[b]$, 0.52 \pm 0.02$[c]$, 0.50$[f]$)$
3	2.19[d]	0.40
		$(0.39 \pm 0.02$[b]$, 0.40$[f]$)$
4	2.25[d]	0.31
		$(0.33$[f]$)$
5	2.33[d]	0.27
		$(0.286$[f]$)$
$6 - \varepsilon$	$\frac{5}{2} - \frac{1}{12}\varepsilon$[e]	$\frac{1}{4} + \frac{5}{168}\varepsilon$ (exact)
		$(\frac{1}{4} + \frac{1}{32}\varepsilon$[f]$)$
≥ 6	$\frac{5}{2}$	$\frac{1}{4}$

[a] Using a conjecture of den Nijs (1979) and its extensions by Pearson (1980) and Nienhuis *et al* (1980).
[b] Gould and Holl (1981).
[c] Family and Reynolds (1981).
[d] Nakanishi and Stanley (1980).
[e] Harris *et al* (1975), Priest and Lubensky (1976), Amit (1976).
[f] Flory theory, $\nu = 2/(d+2)$ (Isaacson and Lubensky 1980).

5. Directed percolation

The upper critical dimension for directed percolation is 5 and in the MF approximation $\theta = \frac{5}{2}$ (Obukhov 1980), $\nu_\parallel = \frac{1}{2}$ and $\nu_\perp = \frac{1}{4}$ (Redner and Yang 1982, Day and Lubensky 1982), which implies that $\alpha = \beta = \gamma = \frac{1}{2}$ in the MF approximation (cf equation (12)). Using these results in (19) we find that $m = 0$ and therefore the hyperscaling relation for directed percolation is modified to $2 - \alpha = \nu_\parallel + \nu_\perp(d-1)$. Since, from (12), $2 - \alpha = \theta - 1$, we find that for directed percolation

$$\theta - 1 = \nu_\parallel + (d-1)\nu_\perp. \tag{21}$$

Exponent θ has only been calculated directly in $d = 2$ (Dhar and Barma 1981), and the exponents $\nu_\parallel$ and $\nu_\perp$ are only known in Flory theory (Redner and Coniglio 1982, Lubensky and Vannimenus 1982). However, in $d = 3$ (Blease 1977a, b, c) and in $d = 5 - \varepsilon$ (Obukhov 1980) γ_p and β_p have been calculated by series expansions and ε expansions respectively. Using the scaling relation (Dhar and Barma 1981) $\theta = (3\beta_p + 2\gamma_p)/(\beta_p + \gamma_p)$ we have determined θ in $d = 3$ and $5 - \varepsilon$ and the results are given in table 2. The percolation exponent $\nu_\parallel^P$, which agrees with the usual definition of $\xi_\parallel$, has been determined by Cardy and Sugar (1980) to first order in $\varepsilon = 5 - d$;

$$\nu_\parallel^P = 1 + \tfrac{1}{12}\varepsilon. \tag{22}$$

Table 2. Directed percolation cluster number exponent θ determined from (21) using values of $\nu_\parallel$ and $\nu_\perp$ given below. Other estimates of θ are given in parentheses.

d	$\nu_\parallel$	$\nu_\perp$	$\theta(=\nu_\parallel + \nu_\perp(d-1)+1)$
2	0.6875^a	0.4375^a	2.125
			(2.112 ± 0.005^b)
3	0.60^a	0.35^a	2.30
			(2.28 ± 0.02^c)
4	0.542^a	0.292^a	2.418
$5-\varepsilon$	$\frac{1}{2} + \frac{1}{24}\varepsilon^d$	$\frac{1}{4} + \frac{1}{32}\varepsilon^d$	$\frac{5}{2} - \frac{1}{12}\varepsilon$
$\geqslant 5$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{2}$

[a] Flory theory, $\nu_\parallel = (d+9)/(4d+8)$, $\nu_\perp = 7/(4d+8)$ (Redner and Coniglio 1982, Lubensky and Vannimenus 1982).
[b] Dhar and Barma (1981).
[c] Blease (1977a, b, c).
[d] $\nu_\parallel = \nu_\parallel^P/(\beta_p + \gamma_p)$ and $\nu_\perp = \nu_\perp^P/(\beta_p + \gamma_p)$ where $\nu_\parallel^P$ is given in Cardy and Sugar (1980), β_p and γ_p are given in Obukhov (1980) and $\nu_\perp^P = (2\beta_p + \gamma_p - \nu_\parallel^P)/(d-1)$.

Using (22) and the ε-expansion results of Obukhov (1980) for β_p and γ_p in $2 - \alpha_p = \gamma_p + 2\beta_p = \nu_\parallel^P + (d-1)\nu_\perp^P$ (Klein and Kinzel 1981) we find

$$\nu_\perp^P = \tfrac{1}{2} + \tfrac{1}{16}\varepsilon. \tag{23}$$

The ε-expansion results for β_p and γ_p and γ_p (Obukhov 1980) together with (22) and (23) may be used to obtain $\nu_\parallel = \nu_\parallel^P/(\beta_p + \gamma_p)$ and $\nu_\perp = \nu_\perp^P/(\beta_p + \gamma_p)$ to first order in $\varepsilon = 5 - d$. The results are

$$\nu_\parallel = \tfrac{1}{2} + \tfrac{1}{24}\varepsilon \qquad \nu_\perp = \tfrac{1}{4} + \tfrac{1}{32}\varepsilon. \tag{24}$$

Once (24) is substituted in the right-hand side of (21), with $d = 5 - \varepsilon$, the result agrees exactly with the ε-expansion result for $\theta - 1$.

Using the Flory theory (Redner and Coniglio 1982, Lubensky and Vannimenus 1982) results $\nu_\parallel = (d + 9)/(4d + 8)$ and $\nu_\perp = 7/(4d + 8)$ in (21) we find $\theta = (6d + 5)/(2d + 4)$. The numerical values of θ in $d = 2$ to 5 are given in table 2 and are found to be in excellent agreement with numerical estimates of θ, where available. It would be very useful to obtain θ directly in $d = 3$ and 4 in order to test the Flory theory and relation (21).

6. Lattice animals

Parisi and Sourlas (1981) have used the connection between lattice animals and the Lee–Yang edge singularity problem to obtain a relation between ν and θ, and to evaluate θ in both $d = 2$ and 3. Here we show that the Parisi–Sourlas relation can be obtained directly without reference to the Lee–Yang problem.

In the MF theory $\theta = \frac{5}{2}$ (see e.g. Stauffer *et al* 1982). Using this result in (12) we find $\alpha = \beta = \gamma = \frac{1}{2}$ in the MF approximation for isotropic lattice animals. The upper critical dimension for lattice animals is 8 and the MF value of ν is $\frac{1}{4}$ (Lubensky and Isaacson 1979, de Gennes 1980). Once these results are substituted in (19), they give $m = -2$, implying the following modified form of hyperscaling relation for isotropic lattice animals:

$$2 - \alpha = \nu(d - 2). \tag{25}$$

Since $2 - \alpha = \theta - 1$ (cf e.g. (12)), this gives

$$\theta - 1 = \nu(d - 2). \tag{26}$$

The exponent θ has been determined numerically up to $d = 8$ using series expansions (see Gaunt (1980) and references therein). We have used these results to determine ν in $d = 3$ to 8 and the results are given in table 3. Note that (26) cannot be used in $d = 2$ to determine ν because $\theta = 1$, and therefore for completeness we have quoted the values of ν obtained from renormalisation group calculations for bond animals (Family 1980) and for site animals (Family 1982, Derrida and DeSeze 1982) in $d = 2$.

7. Directed animals

It has recently been shown by Redner and Yang (1982) and Day and Lubensky (1982) that for directed animals $d_c = 7$ and in MF theory $\theta = \frac{3}{2}$, $\nu_\parallel = \frac{1}{2}$ and $\nu_\perp = \frac{1}{4}$. In addition, Day and Lubensky (1982) have chosen a definition of 'susceptibility' for directed animals in which

$$\gamma = 2 - \theta \tag{27}$$

instead of (12). This choice is useful (Day and Lubensky 1982) because it leads to the usual MF value of $\gamma = \frac{1}{2}$; otherwise it would be $\frac{3}{2}$. Since K still plays the role of both a temperature and a field variable, one has $\alpha = \gamma$, which implies that for directed animals in MF theory $\alpha = \frac{1}{2}$. Using these MF values at $d_c = 7$ in (19) we find that

Table 3. Lattice animal cluster radius exponent ν determined from (26) and values of θ listed below. Other estimates of ν are given in parentheses.

d	θ	$\nu(=(\theta-1)/(d-2))$
2	1^{a} (exact)	—
		$(0.637^{b}, 0.647\pm0.02^{c},$
		$0.6408\pm0.0003^{d})$
3	$\frac{3}{2}^{a}$ (exact)	$\frac{1}{2}$ (exact^{a})
		$(\frac{1}{2}^{f})$
4	1.9^{e}	0.45
		(0.42^{f})
5	2.2^{e}	0.4
		(0.36^{f})
6	2.3^{e}	0.325
		(0.31^{f})
7	2.4^{e}	0.28
		(0.28^{f})
$\geqslant 8-\varepsilon$	$\frac{5}{2}-\frac{1}{12}\varepsilon^{g}$	$\frac{1}{4}+\frac{1}{36}\varepsilon$ (exact^{g})
8	$\frac{5}{2}$	$\frac{1}{4}$

[a] Parisi and Sourlas (1981).
[b] Family (1980).
[c] Family (1982).
[d] Derrida and DeSeze (1982).
[e] Gaunt (1980). (An error of at least ±0.1 should be associated with the values listed above.)
[f] Flory theory, $\nu = 5/(2d+4)$ (Isaacson and Lubensky 1980, Daoud and Joanny 1981).
[g] Lubensky and Isaacson (1979).

$m = -1$, and hyperscaling is modified to

$$2-\alpha = \nu_{\perp}(d-1) \tag{28}$$

so that $\nu_{\perp}$ is related to α in the usual way. Furthermore, substituting $\alpha = 2-\theta$ in (28) we find that for directed animals

$$\theta = \nu_{\perp}(d-1). \tag{29}$$

Recently Dhar *et al* (1982) have conjectured that $\theta = \frac{1}{2}$ in $d = 2$. Using this value in (29) we find $\nu_{\perp} = \frac{1}{2}$ in $d = 2$, in agreement with the bond directed animal series result of Redner and Yang (1982), and the conjecture of Day and Lubensky (1982) that the directed animal exponents may correspond to those of isotropic animals in $d+1$.

In $d = 3$ to 7 we have used the flory results for $\nu_{\perp}$ (Redner and Coniglio 1982, Lubensky and Vannimenus 1982) to obtain $\theta = 9(d-1)/(4d+8)$ and the results are given in table 4. The series expansion results of Redner and Yang (1982) are also listed in table 4. Although the series expansion results of Redner and Yang (1982) are not highly accurate, they are consistent with the values obtained here. Relation (29) can also be shown to be exact in $d = 7-\varepsilon$ dimensions by using the ε-expansion results for θ and $\nu_{\perp}$ (Day and Lubensky 1982).

Day and Lubensky (1982) have conjectured that $\nu_{\perp}$ and θ for directed animals in d dimensions are possibly the same as ν and $\theta-1$ for isotropic animals in $d+1$ dimensions. This conjecture can be readily tested numerically by comparing the values of θ and $\nu_{\perp}$ in d dimensions in table 4 with the values of $\theta-1$ and ν in $d+1$ dimensions

Table 4. Directed lattice animal exponent θ as determined from (29) using values of $\nu_\perp$ listed below. Other estimates of θ are given in parentheses. $\nu_\parallel$ is only given for comparison; it has not yet been related to other exponents.

d	$\nu_\parallel$	$\nu_\perp$	$\theta(=\nu_\perp(d-1))$
2	0.800 ± 0.001^a	0.50 ± 0.003^a	0.50 ± 0.003 $(\frac{1}{2}^b, 0.53 \pm 0.01^a)$
3	0.70^c	0.45^c	0.90 (0.94 ± 0.02^a)
4	0.625^c	0.375^c	1.125 (1.20 ± 0.05^a)
5	0.5714^c	0.3214^c	1.2856 (1.35 ± 0.15^a)
6	0.53125^c	0.28125^c	1.40625 (1.40 ± 0.15^a)
$7-\varepsilon$	$\frac{1}{2} + \frac{1}{24}\varepsilon^d$	$\frac{1}{4} + \frac{1}{36}\varepsilon^d$	$\frac{3}{2} - \frac{1}{12}\varepsilon$ (exact^d)
$\geqslant 7$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{2}$

^{a} Series expansions (Redner and Yang 1982).
^{b} Dhar *et al* (1982) have conjectured that $\theta = \frac{1}{2}$ is exact in $d = 2$.
^{c} Flory theory, $\nu_\parallel = (d+11)/(4d+8)$, $\nu_\perp = 9/(4d+8)$ (Redner and Coniglio 1982, Lubensky and Vannimenus 1982).
^{d} Day and Lubensky (1982).

in table 3. Clearly numerical evidence seems to support their conjecture. A more interesting connection is found from the form of equation (29). By letting $\nu_\perp \to \nu$, $\theta \to \theta - 1$ and $d \to d - 1$ in (29) we find $\theta - 1 = \nu(d-2)$, which is exactly equivalent to relation (26) for isotropic lattice animals.

A remaining problem in directed animals is to relate $\nu_\parallel$ to the other exponents.

Acknowledgments

Note added in proof. After this work was completed we learned of two recent works on directed lattice animals: Stanley *et al* have enumerated both site and bond directed lattice animals for arbitrary dimension and have obtained very accurate values for θ for directed animals in $7 \geqslant d \geqslant 2$ which they use to test a possible relation between the Lee–Yang problem and directed lattice animals and compare with the Flory approximation for θ obtained here; John Cardy has shown that the directed animal problem in d dimensions is equivalent to the Lee–Yang problem in $d - 1$ dimensions and from this correspondence has also obtained relation (29).

References

Als-Nielsen J and Birgeneau R J 1977 *Am. J. Phys.* **45** 554
Amit D J 1976 *J. Phys. A: Math. Gen.* **9** 1441

Blease J 1977a *J. Phys. C: Solid State Phys.* **10** 917
—— 1977b *J. Phys. C: Solid State Phys.* **10** 925
—— 1977c *J. Phys. C: Solid State Phys.* **10** 3461
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Daoud and Joanny 1981 *J. Physique* **42** 1359
Day A R and Lubensky T C 1982 *J. Phys. A: Math. Gen.* **15** L285
Derrida B and DeSeze L 1982 *J. Physique* **43** 475
Dhar D and Barma M 1981 *J. Phys. C: Solid State Phys.* **14** L1
Dhar D, Phani M K and Barma M 1982 *J. Phys. A: Math. Gen.* **15** L279
Essam J W 1980 *Rep. Prog. Phys.* **43** 833
Essam J W and De'Bell K 1981 *J. Phys. A: Math. Gen.* **14** L459
Family F 1980 *J. Phys. A: Math. Gen.* **13** L325
——1982 *Preprint*
Family F and Coniglio A 1980 *J. Phys. A: Math. Gen.* **13** L403
Family F and Reynolds P J 1981 *Z. Phys.* B **45** 123
Fisher M E and Essam J W 1961 *J. Math. Phys.* **2** 609
Gaunt D S, Sykes M F and Ruskin H 1976 *J. Phys. A: Math. Gen.* **9** 1899
Gaunt D S 1980 *J. Phys. A: Math. Gen.* **13** L97
de Gennes P G 1979 *Scaling Concepts in Polymer Physics* (Ithaca, NY: Cornell UP)
——1980 *C. R. Acad. Sci., Paris* **291** 17
Gould H and Holl K 1981 *J. Phys. A: Math. Gen.* **14** L443
Harris A B and Lubensky T C 1981 *Phys. Rev.* B **24** 2656
Harris A B, Lubensky T C, Holcomb W K and Dasgupta C 1975 *Phys. Rev. Lett.* **35** 327
Havlin S and Ben-Avraham D 1982 *J. Phys. A: Math. Gen.* **15** L321
Isaacson J and Lubensky T C 1980 *J. Physique Lett.* **41** L469
Kertész J and Vicsek T 1980 *J. Phys. C: Solid State Phys.* **13** L343
Kinzel W and Yeomans J 1981 *J. Phys. A: Math. Gen.* **14** L163
Klein W and Kinzel W 1981 *J. Phys. A: Math. Gen.* **14** L405
Lubensky T C and Isaacson J 1979 *Phys. Rev.* A **20** 2130
Lubensky T C and Vannimenus J 1982 *J. Physique Lett.* **43** L377
Nakanishi H and Stanley H E 1980 *Phys. Rev.* B **22** 2466
Nienhuis B, Riedel E K and Schick M 1980 *J. Phys. A: Math. Gen.* **13** L189
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Obukhov S P 1980 *Physica* **101A** 145
Parisi G and Sourlas N 1981 *Phys. Rev. Lett.* **46** 871
Pearson R B 1980 *Phys. Rev.* B **22** 2579
Priest R G and Lubensky T C 1976 *Phys. Rev.* B **13** 4159
Redner S and Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** L273
Redner S and Reynolds P J 1981 *J. Phys. A: Math. Gen.* **14** L55
Redner S and Yang Z R 1982 *J. Phys. A: Math. Gen.* **15** L177
Reynolds P J 1980 unpublished
Schlögl F 1972 *Z. Phys.* **253** 147
Stanley H E, Reynolds P J, Redner S and Family F 1982 in *Real-Space Renormalization* ed T Burkhardt
     and J M J van Leeuwen (New York: Springer) p 169
Stauffer D 1979 *Phys. Rep.* **54** 1
Stauffer D, Coniglio A and Adam M 1982 *Adv. Polym. Sci.* **44** 103
Wu F Y and Stanley H E 1982 *Phys. Rev. Lett.* **48** 775